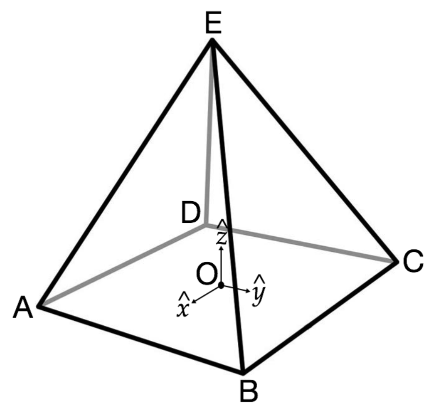
Exam Symmetry in Physics

Date	May 12, 2015
Room	A. Jacobshal 01
Time	18:30 - 21:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- You are not allowed to use the lecture notes, nor other notes or books
- All subquestions (a, b, etc) of the 3 exercises (16 in total) have equal weight
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Exercise 1

Consider a pyramid with a regular square as base with corners labeled by A, B, C, D and its apex E above the center O of the square base (see figure). Its symmetry group is C_{4v}.



(a) Identify all symmetry transformations that leave this pyramid invariant and divide them into conjugacy classes, using geometrical arguments.

(b) Give an identification between elements of C_{4v} and D_4 and argue that the two groups are isomorphic.

(c) Construct the character table of C_{4v} and explain how the entries are obtained.

(d) Construct explicitly the three-dimensional vector representation D^V for the two transformations that generate C_{4v} and check the determinants.

(e) Decompose D^V of C_{4v} into irreps and use this to conclude whether this group in principle allows for an invariant three-dimensional vector, such as an electric dipole moment.

(f) Do the same, i.e. answer questions (d) and (e), for the axial-vector representation D^A, and conclude whether the group in principle allows for an invariant three-dimensional axial-vector, such as a magnetic dipole moment.

(g) Determine the characters of the direct product representation $D^V \otimes D^V$ of C_{4v} and use them to determine the number of independent invariant tensors T^{ij} $(i, j = 1, 2, 3)$ (no need to construct them explicitly).

Exercise 2

Consider a non-isotropic medium with a conductivity tensor $\sigma_{ij} \neq \delta_{ij}$ in three-dimensions ($i, j = 1, 2, 3$). When exposed to an electric field $\vec{E}$, there will be an electrical current $\vec{j}$ in the medium, according to $j_i = \sigma_{ij} E_j$ (here and below summation over repeated indices is implicit).

(a) Use the transformation properties of the equation $j_i = \sigma_{ij} E_j$ to derive that σ_{ij} transforms into $\sigma'_{kl} = D^V_{ki} D^V_{lj} \sigma_{ij}$ under (subgroups of) rotations.

(b) Show that σ_{ij} is invariant if it satisfies $\sigma D^V = D^V \sigma$.

(c) Explain why in the case of $SO(3)$ symmetry, in other words for an isotropic medium, the only invariant tensor is proportional to the identity.

(d) Write down the most general form of an invariant σ tensor for a medium that has an $SO(2)$ symmetry around the $\hat{z}$-axis.

(e) Explain why the trace of σ_{ij} transforms as a scalar.

(f) Use $j_i = \sigma_{ij} E_j$ to explain how $\epsilon_{ijk} \sigma_{jk}$ transforms under $O(3)$ transformations.

Exercise 3

Consider the group $SO(2)$ of rotations around the $\hat{z}$-axis in three dimensions. Consider its action on the angular momentum states $|l, m\rangle$ through the operator

$$U(R_z) = \exp\left(-\frac{i}{\hbar}\theta\, L_z\right), \tag{1}$$

where $R_z = R_z(\theta)$ is a rotation over an angle θ around the $\hat{z}$-axis.

(a) Write down the explicit matrix for L_z acting on the space of $|2, m\rangle$ states.

(b) Write down the explicit matrix $D^{(l)}(R_z)$ for $U(R_z)$ with $l = 2$.

(c) Show that $D^{(l=2)}(R_z) \in U(5)$ and explain how it is related to $SO(2)$.